

*Saturday, October 26, 2024*

**Problem 1.** There are 1001 stacks of coins  $S_1, S_2, \dots, S_{1001}$ . Initially, stack  $S_k$  has  $k$  coins for each  $k = 1, 2, \dots, 1001$ . In an operation, one selects an ordered pair  $(i, j)$  of indices  $i$  and  $j$  satisfying  $1 \leq i < j \leq 1001$  subject to two conditions:

- The stacks  $S_i$  and  $S_j$  must each have at least one coin.
- The ordered pair  $(i, j)$  must *not* have been selected in any previous operation.

Then, if  $S_i$  and  $S_j$  have  $a$  coins and  $b$  coins, respectively, one removes  $\gcd(a, b)$  coins from each stack.

What is the maximum number of times this operation could be performed?

**Problem 2.** Let  $k$  be a fixed positive integer. For each integer  $1 \leq i \leq 4$ , let  $x_i$  and  $y_i$  be positive integers such that their least common multiple is  $k$ . Suppose that the four points  $(x_1, y_1)$ ,  $(x_2, y_2)$ ,  $(x_3, y_3)$ ,  $(x_4, y_4)$  are the vertices of a non-degenerate rectangle in the Cartesian plane. Prove that  $x_1 x_2 x_3 x_4$  is a perfect square.

**Problem 3.** Let  $ABC$  be an acute triangle with incenter  $I$ . Two distinct points  $P$  and  $Q$  are chosen on the circumcircle of  $ABC$  such that

$$\angle API = \angle AQI = 45^\circ.$$

Lines  $PQ$  and  $BC$  meet at  $S$ . Let  $H$  denote the foot of the altitude from  $A$  to  $BC$ . Prove that  $\angle AHI = \angle ISH$ .



The organizers of the USEMO are grateful to be sponsored this year by the **CoRe Lab, Institute of Artificial Intelligence, Peking University**.

*Time limit: 4 hours and 30 minutes.  
Each problem is worth seven points.*

*Sunday, October 27, 2024*

**Problem 4.** Find all sequences  $a_1, a_2, \dots$  of nonnegative integers such that for all positive integers  $n$ , the polynomial

$$1 + x^{a_1} + x^{a_2} + \dots + x^{a_n}$$

has at least one integer root. (Here  $x^0 = 1$ .)

**Problem 5.** Let  $ABC$  be a scalene triangle whose incircle is tangent to  $BC, CA, AB$  at  $D, E, F$  respectively. Lines  $BE$  and  $CF$  meet at  $G$ . Prove that there exists a point  $X$  on the circumcircle of triangle  $EFG$  such that the circumcircles of triangles  $BCX$  and  $EFG$  are tangent, and

$$\angle BGC = \angle BXC + \angle EDF.$$

**Problem 6.** Let  $n$  be an odd positive integer and consider an  $n \times n$  chessboard of  $n^2$  unit squares. In some of the cells of the chessboard, we place a knight. A knight in a cell  $c$  is said to *attack* a cell  $c'$  if the distance between the centers of  $c$  and  $c'$  is exactly  $\sqrt{5}$  (in particular, a knight does not attack the cell which it occupies).

Suppose each cell of the board is attacked by an even number of knights (possibly zero). Show that the configuration of knights is symmetric with respect to all four axes of symmetry of the board (i.e. the configuration of knights is both horizontally and vertically symmetric, and also unchanged by reflection along either diagonal of the chessboard).



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